What Community College Developmental Mathematics Students Understand about Mathematics

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The nation is facing a crisis in its community colleges: more and more students are attending community colleges, but most of them are not prepared for college-level work. The problem may be most dire in mathematics. By most accounts, the majority of students entering community colleges are placed (based on placement test performance) into "developmental" (or remedial) mathematics courses (e.g., Adelman, 1985; Bailey et al., 2005). The organization of developmental mathematics differs from school to school, but most colleges have a sequence of developmental mathematics courses that starts with basic arithmetic, then goes on to pre-algebra, elementary algebra, and finally intermediate algebra, all of which must be passed before a student can enroll in a transfer-level college mathematics course.

Because the way mathematics has traditionally been taught is sequential, the implications for students who are placed in the lower-level courses can be quite severe. A student placed in basic arithmetic may face two full years of mathematics classes before he or she can take a college-level course. This might not be so bad if they succeeded in the two-year endeavor. But the data show that most do not: students either get discouraged and drop out all together, or they get weeded out at each articulation point, failing to pass from one course to the next (Bailey, 2009). In this way, developmental mathematics becomes a primary barrier for students ever being able to complete a post-secondary degree, which has significant consequences for their future employment.

One thing not often emphasized in the literature is the role that our K-12 education system plays in this problem. We know from international studies that U.S. mathematics education produces student achievement scores that fall below the scores of students in most other industrialized nations. But the fact that community college students, most of whom graduate from U.S. high schools, are not able to perform basic arithmetic, pre-algebra, and algebra, shows the real cost of our failure to teach mathematics in a deep and meaningful way in our elementary, middle, and high schools. Although our focus here is on the community college students, it is important to acknowledge that the methods used to teach mathematics in K-12 schools are not succeeding, and that the limitations of students' mathematical proficiency are cumulative and increasingly obvious over time.

The limitations in K-12 teaching methods have been well-documented in the research literature. The Trends in International Mathematics and Science Study (TIMSS) video studies (Stigler & Hiebert, 1999; Hiebert et al., 2003) showed that the most common teaching methods used in the U.S. focus almost entirely on practicing routine procedures, with virtually no emphasis on understanding of core mathematics concepts that might help students forge connections among the numerous mathematical procedures that make up the mathematics curriculum in the U.S. The high-achieving countries in TIMSS, in contrast, use instructional methods that focus on actively engaging students with understanding mathematical concepts. Procedures are taught, of course, but are connected with the concepts on which they are based. In the U.S., procedures are more often presented as step-by-step actions that students must memorize, not as moves that make sense mathematically.

Given that U.S. students are taught mathematics as a large number of apparently-unrelated procedures that must be memorized, it is not surprising that they forget most of them by the time they enter the community college. It is true that some students figure out, on their own, that mathematics makes sense and that procedures forgotten can be reconstructed based on a relatively small number of core concepts. And even a few students who don't figure this out are smart enough to actually remember the procedures they are taught in school. But many students don't figure this out, and these are the ones that swell the ranks of students who fail the placement tests and end up in developmental mathematics.

Sadly, all the evidence we have (which is not much) shows that although community college faculty are far more knowledgeable about mathematics than are their K-12 counterparts (Lutzer et al., 2007), their teaching methods may not differ much from those used in K-12 schools (Grubb, 1999). "Drill-and-skill" is still thought to dominate most instruction (Goldrick-Rab, 2007). Thus, students who failed to learn how to divide fractions in elementary school, and who also probably did not benefit from attempts to re-teach the algorithm in middle and high school, are basically presented the same material in the same way yet again. It should be no surprise that the methods that failed to work the first time also don't work in community college. And yet that is the best we have been able to do thus far.

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Currently there is great interest in reforming developmental mathematics education at the community college. Yet, it is worth noting that almost none of the reforms have focused on actually changing the teaching methods and routines that define the teaching and learning of mathematics in community colleges. Many schools have instituted courses that teach students how to study, how to organize their time, and how to have a more productive motivational stance towards academic pursuits (Zachry, 2008; Zeidenberg et al, 2007). They have tried to make it easier for students burdened with families and full-time jobs to find time to devote to their studies. They have created forms of supplemental instruction (Blanc et al., 1983; Martin & Arendale, 1994) and learning assistance centers (Perin, 2004). They have tried to break down bureaucratic barriers that make it difficult for students to navigate the complex pathways through myriad courses that must be followed if students are ever to emerge from developmental math and pass a transfer-level course. Some have redesigned the curriculum; they've accelerated it, slowed it down, or tried to weed out unnecessary topics (e.g., Lucas & McCormick, 2007). Yet few have questioned the methods used to teach mathematics (Zachry, 2008).

An assumption we make in this report is that substantive improvements in mathematics learning will not occur unless we can succeed in transforming the way mathematics is taught. In particular, we are interested in exploring the hypothesis that these students who have failed to learn mathematics in a deep and lasting way up to this point might be able to do so if we can first convince them that mathematics makes sense, and then provide them with the tools and opportunities to think and reason. In other words, if we can teach mathematics as a coherent and tightly related system of ideas and procedures that are logically linked, might it not be possible to accelerate and deepen students' learning and create in them the disposition to reason about fundamental concepts? Might this approach reach those students who have not benefited from the way they have been taught mathematics up to this point? (English & Halford, 1995).

Consideration of this hypothesis led us to inquire into what we actually know about the mathematics knowledge and understanding of students who are placed into developmental math courses. Surprisingly, an extensive search of the literature revealed that we know almost nothing about these aspects of community college students. Grubb (2005) made a similar point: we know quite a bit about community college teachers and about the institutions in which they work

> …but our knowledge of students and their attitudes toward learning is sorely lacking. …The conventional descriptions of developmental students stress demographic characteristics (for example, first-generation college status and ethnicity) and external demands (such as employment and family), but aside from finding evidence of low self-esteem and external locus of control, there has been little effort to understand how developmental students think about their education. (Grubb & Cox, 2005, p. 95).

Most of what we know about the mathematical knowledge of community college students we learn from placement tests (Accuplacer, Compass, MDTP). But placement test data is almost impossible to come by due to the high-stakes nature of the tests and the need to keep items protected. Further, the most commonly used tests (Accuplacer and Compass) are adaptive tests, meaning that students take only the minimum number of items needed to determine their final score, and so they don't take items that might give a fuller picture of their mathematical knowledge. Finally, most of the items on the placement tests are procedural in nature: they are designed to assess what students are able to do, but not what students understand about fundamental mathematical concepts.

Because of this gap in the literature, we undertook the study reported here. Our aim was to gather information about what students actually understand about the mathematics that underlie the topics they've been taught, including their understanding of the reasons for using known procedures. We also sought, specifically, evidence that students used reasoning in answering mathematical questions.[1]

We investigated these questions using two sources of data. The first data source was one collection of the placement tests to which we've referred, those developed by the Mathematics Diagnostic Testing Project (MDTP). The purpose of examining it was to see what we could glean about student understanding from an existing measure. The MDTP tests are unusual in that they are not a commercially designed or administered test, and are not adaptive. They were developed back in the early 1980s as a joint CSU/UC project whose members included mathematics faculty from all segments of public education in California. Specifically, the group included mathematics professors from public institutions of higher education and high school mathematics teachers, as well as some physical sciences higher education faculty. The goal of the first test was not only placement of entering University students, but also to give feedback to high schools on how well prepared their students were in mathematics areas critical for later success in college. But many community colleges do use the MDTP for placement purposes, including more than 50 in California. Interestingly, the tests used by most California community colleges—in particular, the test used in this study—have not changed since 1986. For this study we have been able to get access to all placement test data given by Santa Barbara City College for the past nearly 20 years. For the present report, we will present findings from the tests administered during the 2008-2009 academic year.

The second data source was a survey of math questions that we administered to a convenience sample of 748 community college developmental mathematics students. There were a total of twelve questions, and each student answered four. The purpose of this survey was to delve more deeply into students' thinking and to gather information that might help us in the design one-on-one interviews with students (the results of which are forthcoming).

More details on methods will be presented together with results in the following sections.

Placement Test Data

Participants and Tests

All Santa Barbara City College students who took the Mathematics Diagnostic Testing Project (MDTP) placement tests during the 2008-2009 school year were included in the study. Tests were administered at three time points during the year: summer and fall of 2008, and spring of 2009. In all, 5830 tests were administered.

There were four different tests: Algebra Readiness, Elementary Algebra, Intermediate Algebra, and Pre-Calculus. Although the majority of students took only one test, some took more than one in order to determine their placement in the mathematics sequence. The gender of participants was relatively stable across tests, with slightly more males than females in each case. Ethnicity varied somewhat depending on test form, with the Hispanic and Black populations decreasing as test level increased. The Asian population increased as test level increased. Age decreased slightly with increase in test level.

There are 50 multiple choice items on the Algebra Readiness and Elementary Algebra assessments, 45 on the Intermediate Algebra assessment, and 60 on the Pre-Calculus assessment. The items on each assessment are grouped into multiple subscales, defined by the test writers. For the Algebra Readiness, Elementary Algebra, and Intermediate Algebra assessments, students had 45 minutes to complete the test. For the Pre-Calculus test students were allowed 90 minutes.

Student Difficulties

The examination of standardized test results often begins and ends with an analysis of mean scores. Our primary interest in the MDTP, however, lay not in the percent of items students got correct on the test or on a subscale of it, but rather in what their answer selections could tell us about their thinking. A correctly chosen response on a multiple choice test *may* indicate understanding. (That's an issue to be pursued with the forthcoming interviews.) The selection of a wrong answer can sometimes be even more telling. Students occasionally answer questions randomly, but more often than not, they make their selection with some thought. Exploring the patterns of students selections of wrong answers was therefore our starting point in identifying student difficulties.

[1] We differentiate between students' understanding of the reasons for executing particular procedures from their ability to reason. The former reflects their understanding of the link between procedures and the mathematical concepts that underlie them and the latter reflects their ability to reach a logical conclusion based on what they know.

Our examination of incorrect answers has focused thus far on the Algebra Readiness and Elementary Algebra assessments. For each we determined which items on the test proved most difficult for students. There were three criteria upon which our definition of difficulty was based. First, we included all items for which fewer than 25 percent of participants marked the correct answer. We also included items for which more students selected one incorrect answer than selected the correct answer. Finally, we counted those items for which there were two incorrect answer options, each of which was selected by at least 20 percent of students. The result was a collection of 13 difficult items for Algebra Readiness and 10 difficult items for Elementary Algebra. Those items and the common errors made on them are described in Tables 1 and 2, respectively. It is important to note that the table describes common procedural errors. Errors in reasoning are described in a subsequent section.

Table 1. Difficult items on the Algebra Readiness form of the MDTP (in ascending order of percent correct), 2008-2009.

Item description	% of students who answered correctly	Common error(s)	% of students who made common error
Add a simple fraction and a decimal.	19	Found GCF	28
Find LCM of two numbers.	21	Converted decimal to a fraction, then added numerators and added denominators	59
Order four numbers (two simple fractions and two decimals).	22	Converted fractions to decimals and ordered by number of digits	36
		Represented 1/3 as 0.3 and ordered decimals by number of digits	24
Add two squares under a radical.	23	Added two squares, but failed to take the square root, stopping short of solving	31
		Assumed $a^2 + b^2 = (a + b)^2$ or that $\sqrt{a^2+b^2} = \sqrt{a^2} + \sqrt{b^2}$	25
Find a missing length for one of two similar triangles.	24	Approximated ratio	25
		Multiplied two bases of one triangle and divided by a base of the second triangle	23
Add two improper fractions.	24	Added numerators and added denominators	41
Find the missing value of a portion of a circle that has two portions labeled with simple fractions	26	Added numerators and denominators of the two fractions provided, stopping short of solving [other option also involved stopping short]	45
Find the diameter of a circle, given the area.	26	Found radius and failed to cancel π, stopping short of solving	37
Find the percent increase between two dollar amounts.	27	Found dollar amount increase and labeled it as a percentage, stopping short of solving	43
		Used larger of the two amounts as denominator when calculating increase	23

Find area of half of a square drawn on a coordinate plane.	33	Found area of the square, stopping short of solving	28
Find the largest of four simple fractions.	33	Found smallest fraction or converted to decimals and chose the only fraction that didn't repeat	44
Multiply two simple fractions.	37	Simplified incorrectly before multiplying	22
		Simplified incorrectly before multiplying	20
Divide one decimal by another.	41	Misplaced decimal (omitted zero as a placeholder)	23
		Divided denominator by numerator and misplaced decimal	20

Table 2. Difficult items on the Elementary Algebra form of the MDTP (in ascending order of percent correct), 2008-2009.

Item description	% of students who answered correctly	Common error(s)	% of students who made common error
Add two fractions that include variables	15	Added numerators and added denominators	34
Multiply two fractions that include variables.	16	Simplified incorrectly before multiplying and misplaced negative sign	24
		Simplified incorrectly before multiplying	23
Solve for x in a quadratic equation.	17	Factored the quadratic equation incorrectly and perhaps also solved for x incorrectly	21
Simply a fraction that includes variables.	19	Simplified incorrectly	31
Find the percent that a larger number is of a smaller.	26	Divided the smaller number by the larger and converted the quotient to a decimal	25
		Divided the larger number by the smaller number, but failed to move the decimal in converting to a percent, stopping short of solving	24
Find the value of a number with a negative exponent.	26	Ignored the negative sign in the exponent	50
Find the area of a triangle within a square that shares a common base.	31	Found the area of a triangle different from the one asked	23
		Multiplied the two lengths provided	22

Square a binomial difference.	32	Squared each term in the expression and made an error with the negative sign	38
		Squared each term in the expression and omitted the '*xy*' term	21
Find the difference between two square roots.	34	Subtracted one number from the other and took square root of the difference	24
		Factored the numbers provided and placed the common factor outside the radical without taking its square root	23
Identify the smallest of three consecutive integers given the sum of those integers	46	Constructed equation incorrectly	23

When we examined the errors students made on the most difficult test items, a few core themes emerged. Students' tendencies to make those errors were quite consistent: when they could make these errors, they did. We then looked at the ten items on each of the two tests that were answered correctly by the most students. None of these items provided opportunities for making the kinds of errors we saw among the difficult items. Several of the most common errors involved working with fractions. Across the two placement tests, the most common mistake was to simplify incorrectly. On the Algebra Readiness assessment, two of the frequent errors on difficult problems were caused by simplifying proper fractions incorrectly (e.g., simplifying 9/16 as 3/4).[2] On the Elementary Algebra assessment, three of the frequent errors on difficult problems were made when simplifying terms with variables (e.g., simplifying $(x + 1)/(x^2 + 5)$ as $1/(x + 4)$. In these cases the option chosen showed that either the students factored expressions incorrectly or made no attempt to use factoring.

It was also the case, as is common with younger students, that our community college sample frequently added across the numerator and across the denominator when adding fractions (e.g., 1/2 + 2/3 = 3/5). Three of the commonly chosen wrong answers we examined were caused by that mistake on the Algebra Readiness test and the process presented itself also on the Elementary Algebra assessment. Finally, the Algebra Readiness test also showed multiple instances of converting a fraction to a decimal by dividing the denominator by the numerator (e.g., 5/8 = 8 ÷ 5). These errors reveal that rather than using number sense, students rely on a memorized procedure, only to carry out the procedure incorrectly or inappropriately.

[2] In order to protect the items that appear on the MDTP, items are discussed in general terms and numbers have been changed.

Answer choices related to decimals lead us think that students may not have a firm grasp of place value. For instance, two frequently chosen answer options suggested that students believed that the size of a value written in decimal form was determined by the number of digits in it (e.g., 0.53 < 0.333).

Another emergent theme suggested that students do not know how to operate with exponents and square roots. For example, some students added terms that shared a common exponent (e.g., $4^2 + 5^2 = 9^2$). Others treated the sum of two numbers as the same as the square of those numbers.

Two final themes were related not as much to procedural misunderstanding as they were to problem solving. It was common, particularly on the Algebra Readiness assessment, for students to respond to a multi-step problem by completing only the first step. It was as if they knew the steps to take, but when they saw an intermediate response as an answer option, they aborted the solution process. This may provide evidence that students have a disposition to treat the goal of mathematical problems as getting answers quickly rather than correctly and with understanding. Another possible interpretation is that the student knew the first step, and then knew there was some next step, but couldn't remember it and chose the option matching what s/he knew was correct. "Stopping short" could be used to explain five of the common errors on difficult Algebra Readiness items and one error on a difficult Elementary Algebra item.

Lastly, it appeared as though students sometimes fell back on their knowledge of how math questions are typically posed. It was as if the item (or answer options) prompted their approach to it. For instance, when asked to find the least common multiple of two numbers that also had a greatest common factor other than one, they selected the answer that represented the greatest common factor. For example, if asked for the least common multiple of 6 and 9, students answered 3 (the greatest common factor) instead of 18 (the correct answer). Rarely do students practice finding least common multiples on anything but numbers *without* common factors, so they assumed in this case that the question was actually seeking the greatest common factor.

Students also fell back on what they're typically asked to do when they were presented with a percentage to calculate. Instead of finding what percentage 21 is of 14 (as was asked), they calculated the percentage 14 is of 21. The latter, with a result less than 100 percent, is the more frequent form of the question. Finally, on a geometry problem that prompted students to find the area of a figure, they operated on the values provided in the problem without regard to whether the values were the appropriate ones. They simply took familiar operations and applied them to what was there.

Do We See Evidence of Reasoning?

As with many standardized mathematics tests, the items of the MDTP focus on procedural knowledge; very little reasoning is called for. Because of that, it is difficult to assess reasoning from test scores. When we examine frequent procedural errors though, we can see many cases where, had students reasoned at all about their answer choice, they wouldn't have made the error. This lack of reasoning was pervasive. It was apparent on both the Algebra Readiness and the Elementary Algebra tests, across math subtopics, and on both "easy" and "difficult" items. We will provide a number of specific examples.

On the Elementary Algebra test, students were asked to find the decimal equivalent of an improper fraction. Only one of the available answer options was greater than 1, yet nearly a third of students (32 percent) selected a wrong answer. If students had had a sense for the value of the improper fraction (simply that it represented a number greater than 1) and then scanned the options, they could have eliminated all four distractors immediately and without doing a calculation of any kind.

Another item prompted students to subtract a proper fraction (a value nearly 1) from an improper fraction (a familiar form of one and a half). Again, if students had examined the fractions and developed a sense of what the answer should be, they would have known that it would be slightly more than a half. Surprisingly, 13 percent of students chose a negative number as their answer, revealing that they could not detect that the first fraction was greater than the second.

A geometry problem asked students to find one of the bases of a right triangle, given the lengths of the other two sides. Nearly a quarter of students selected an answer that was geometrically impossible. They selected lengths that could not have made a triangle, given the two lengths provided. Two of their answer choices yielded triangles with two sides whose sum was equal to the length of the third side. The third choice produced a triangle with a base longer than the hypotenuse.

Another geometry problem provided a diagram of similar triangles and asked students to identify a missing length. The base of the larger triangle (*AB* in Figure 1) was indicated to be 28. Students were to use the values of the other sides to find the length of *AC*. One of the answer options was strikingly out of range. Thirteen percent of students said that the length of *AC* was 84. What they did was notice that the length of one of the bases was three times the other and therefore multiplied 28 (i.e., the length of *AB*) by 3 to get their answer. Presumably, they didn't check to see if their answer made logical sense.

Figure 1. Line segments *AC* and *AB* represent the bases of two similar triangles.

So is it the case that students are incapable of reasoning? Are they lacking the skills necessary to estimate or check their answers? In at least one case, we have evidence that community college students have the skills they need. On one Elementary Algebra test item, students were provided values for *x* and *y* and were asked to find the value of an expression in which they were used. (Though the expression included a fraction, there was no need for either simplification or division, two error-prone tasks.) The item proved to be the third easiest on the test, with nearly three quarters of students answering correctly. Their performance on the item demonstrates that they are capable of substituting values and using basic operations to solve. That skill would have eliminated a great number of frequently chosen wrong answers *if students had thought to use it*. If students had only chosen a value for the variable and substituted this value into both the original expression and their answer choice, they could have caught the mistakes they'd made doing such things as executing operations and simplifying. Some people may think of substituting answer options into an item prompt as purely a test-taking strategy, but we argue that verification is a form of reasoning. In this case, it shows that the student knows the two expressions are meant to be equivalent, and should therefore have the same value.

We noted in the introduction that students are taught mathematics as a large number of apparently-unrelated procedures that must be memorized. It appears from the MDTP that the procedures are memorized in isolated contexts. The result is that a memorized procedure isn't necessarily called upon in a novel situation. Procedures aren't seen as flexible tools – tools useful not only in finding, but also in checking answers. What do students think they are doing when they simplify an algebraic expression, or for that matter simplify a fraction? Do they understand that they are generating an equivalent expression or do they think they are merely carrying out a procedure from algebra class?

We cannot know from the MDTP the degree to which students are capable of reasoning, but we do know that their reasoning skills are being underutilized and that their test scores would be greatly improved if they had a disposition to reason.

Survey

Study Participants

Students were recruited from four community colleges in the Los Angeles metropolitan area. All were enrolled in 2009 summer session classes, and all were taking a developmental mathematics class. There were 748 participants (82 in Arithmetic, 334 in Pre-Algebra, 319 in Elementary Algebra, and 13 for whom class information was missing). We collected no data from Intermediate Algebra students, even though it, too, is not a college-credit-bearing class. Our sample lies mainly in the two most common developmental placements: Pre-Algebra and Elementary Algebra.

We asked students to tell us how long it had been since their last math class and the results are shown in Table 3.

Table 3. Length of time since survey study participants' most recent math class.

Time Since Last Math Class	N
1 year or less	346
2 years	118
3-5 years	83
More than 5 years	149
Missing Data	52

Although the modal student in our sample was 20 years old (M = 22.6, SD = 7.3), the histogram of ages has a rather long tail out to the right, with a number of students in their 30s and 40s.

Survey Items

To construct the survey, we began by listing key concepts in the mathematics curriculum, from arithmetic through elementary algebra. They included comparisons of fractions, placement of fractions on a number line, operations with fractions, equivalence of fractions/decimals/percents, ratio, evaluation of algebraic expressions, and graphing linear equations. Survey items were created to assess each of those concepts. To better understand students' thinking, several of the items also included the question, "How do you know?"

The initial survey consisted of 12 questions divided into three forms of four questions each. Each student was randomly given one of the three forms.

Understanding of Numbers and Operations

The first items we will examine tried to get at students' basic understanding of numbers, operations, and representations of numbers. We focused on fractions, decimals, and percents.

In one question students were instructed: "Circle the numbers that are equivalent to 0.03. There is more than one correct response." The eight choices they were asked to evaluate are shown in Table 4, along with the percentage of students who selected each option. (The order of choices has been re-arranged according to their frequency of selection.)

Table 4. Survey question: Circle the Numbers Equivalent to 0.03.

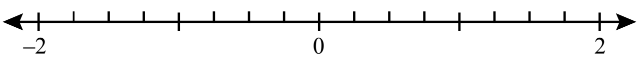
Response Option	Percent of Students Who Marked It as Equivalent to 0.03
3/100	67*
3%	53*
0.030	38*
3/10	23
0.30%	12
30/1000	9*
0.30	6
3/1000	3

*indicates a correct option

Only 4 percent of the students got all answers correct. The easiest two options (3/100 and 3%) were correctly identified by only 67 percent and 53 percent of the students, respectively. It appeared that as the answers departed further from the original form (0.03) students were less likely to see the equivalence. Interestingly, only 9 percent of students correctly identified 30/1000 as equivalent, even though 38 percent correctly identified 0.030. It appears that some students learned a rule (adding a zero to the end of a decimal doesn't change the value), yet only some of these saw that 0.030 was the same as 30/1000. Students clearly are lacking a basic fluency with the representations of decimals, fractions, and percents. The students enrolled in Elementary Algebra did significantly better on the item than those enrolled in Pre-Algebra or Arithmetic ($F(156, 2) = 7.290$, $p = .001$). Yet, even of the students in Algebra, only 17 percent correctly chose 30/1000 as equivalent to 0.03.

Another question asked students to mark the approximate position of two numbers (– 0.7 and 1 3/8) on a number line that ranged from – 2 to +2 (see Figure 2).

Figure 2. Number line on which students were to place -0.7 and 1 3/8.



Only 21 percent of students were able to place both numbers correctly. Thirty-nine percent correctly placed – 0.7, and 32 percent correctly placed 1 3/8. Algebra students performed significantly better than the Arithmetic students ($F(362, 2) = 5.056$, $p < .01$), but still, only 30 percent of Algebra students marked both positions correctly.

On another question students were asked, "If n is a positive whole number, is the product $n \times 1/3$ greater than n, less than n, equal to n, or is it impossible to tell?" Only 30 percent of students selected the correct answer (i.e., less than n). Thirty-four percent said that the product would be greater than n (assuming, we think, that multiplication would always results in a larger number). Eleven percent said the product would be equal to n, and 26 percent said that they could not tell (presumably because they think it would depend on what value is assigned to n).

Interestingly, students in Algebra were no more successful on this question than were students in either of the other two classes ($F(176, 2) = 2.020$, $p < 0.136$). Furthermore, students who reported longer time since their last math class (i.e., 2 years ago) actually did better than students who had studied mathematics more recently (i.e., a year or less ago; $F(166, 3) = 3.139$, $p = 0.027$). This kind of question is not typical of what students would confront in a mathematics class; they are not asked to calculate anything, but just to think through what the answer might be. Perhaps the longer students have been away from formal mathematics classes, the less likely they are to remember what they are supposed to do, and the more they must rely on their own understanding to figure out how to answer a question like this one.

Do We See Evidence of Reasoning?

As we analyzed the students' responses, we started to feel that, first, students will whenever possible just fire off some procedure that they seem to have remembered from before, and, second, that they generally don't engage in reasoning at all, unless there is just no option. When they do reason they have difficulty. No doubt this is due in part to the fragile understanding of fundamental concepts that they bring to the task. But also it indicates a conception of what it means to do mathematics that is largely procedural, and thus a lack of experience reasoning about mathematical ideas.

We asked students, "Which is larger, 4/5 or 5/8? How do you know?" Seventy-one percent correctly selected 4/5 and 24 percent selected 5/8. (Four percent did not choose either answer.) Twenty-four percent of the students did not provide any answer to the question, "How do you know?" Those who did answer the question, for the most part, tried whatever procedure they could think of that could be done with two fractions. For example, students did everything from using division to convert the fraction to a decimal, to drawing a picture of the two fractions, to finding a common denominator. What was fascinating was that although any of these procedures could be used to help answer the question, students using the procedures were almost equally split between choosing 4/5 or choosing 5/8. This was often because they weren't able to carry out the procedure correctly, or because they weren't able to interpret the result of the procedure in relation to the question they were asked. Only 6 percent of the students produced an explanation that did not require execution of a procedure: they simply reasoned that 5/8 is closer to half, and 4/5 is closer to one. No one who reasoned in this way incorrectly chose 5/8 as the larger number.

We asked a related question to a different group of students: "If a is a positive whole number, which is greater: $a/5$ or $a/8$?" If one is reasoning, then this should be an easier question than the previous one. Yet, it proved harder, perhaps because many of the procedures students used to answer the previous question could not be immediately executed without having a value for a. Only 53 percent of our sample correctly chose $a/5$ as the larger number. Twenty-five percent chose $a/8$, and 22 percent did not answer.

We followed up this question by asking, "How do you know?" This time, 36 percent were not able to answer this question. (Interestingly, this percentage was approximately the same for students who chose $a/5$ as for those who chose $a/8$.) Of those who did produce an answer, most could be divided into three categories.

Some students simply cited some single aspect of the two fractions as a sufficient explanation. For example, 5 percent simply said that "8 is bigger" or "8 is the larger number." All of these students incorrectly chose $a/8$ as the larger number. In a related explanation, 17 percent mentioned the denominator as being important - which it is, of course - but half of these students incorrectly chose $a/8$ as the larger number.

Another group of students (10 percent) used a procedure, something they had learned to do. For example, some of them substituted a number for a and then divided to find a decimal, but not always the correct decimal. Others cross multiplied, ending up with $8a$ and $5a$, or found a common denominator (40ths). Approximately half the students who executed one of these procedures chose $a/5$ as larger, and half chose $a/8$. They would execute a procedure, but had a hard time linking the procedure to the question they had been asked to answer.

The most successful students (15 percent) produced a more conceptual explanation. Some of these students interpreted the fractions as division. For example, they pointed out that when you divide a number by five you get a larger number than if you divide it by eight. Others drew pictures, or talked about the number of "pieces" or "parts" a was divided into. Some said that if you "think about a pizza" cut into five pieces vs. eight pieces, the five pieces would be larger. Significantly, all of the students who used these more conceptual explanations correctly chose $a/5$ as the larger number.

Four percent of students said that it was impossible to know which fraction was larger "because we don't know what a is."

We know from previous research that it is difficult for students to make the transition to algebra, to learn to think with variables about quantities. These results, of much older students, suggest that the lack of experience thinking algebraically may actually impede students' understanding of basic arithmetic.

Another test item that revealed students' ability to reason was the following: "If $a + b = c$, which of the following equations are also true? There may be more than one correct response." The possible responses, together with the percentage of students who chose each response, are presented in Table 5.

Table 5. Response options and percent of students choosing them in answer to the question, "If $a + b = c$, which of the following equations are also true?"

Response Option	Percent Students Choosing
$b + a = c$	91*
$c = a + b$	89*
$c - b = a$	45*
$c - a = b$	41*
$b - c = a$	17
$a + b - c = 0$	28*
$c - a + b = 0$	9

*indicates a correct option

Most of the students knew that the first two options were equivalent to $a + b = c$. They knew that the order didn't matter ($a + b = b +$ a) and they knew that you could switch what was on each side of the equals sign without affecting the truth of the equation. Still, 10 percent of students did not know these two things.

It proved much harder for students to recognize that if $a + b = c$, then $c - a = b$ (or, $c - b = a$), with only 45 percent and 41 percent of the students choosing each of these options. Students could have arrived at these two answers either by executing a procedure (e.g., subtracting b from both sides of the equation) or by understanding the inverse relationship between addition and subtraction.

It is illuminating to look at the patterns of response students gave to the following three options:

$c - a = b$
$c - b = a$
$b - c = a$

Even though 40+ percent of students correctly chose the first two options, fully 13 percent chose all three options as correct. This finding suggests that students are examining each option in comparison to the original equation ($a + b = c$), but not necessarily looking at the options compared with each other. It is hard to imagine how someone could believe that the latter two options are simultaneously true, unless they mistakenly think that the order of subtraction ($c - b$ vs. $b - c$) is not important, overgeneralizing the commutative property of addition to apply to subtraction, as well. Only 25 percent of the sample correctly chose both of the first two options but not the third.

A similar analysis can be done with the last two options:

$a + b - c = 0$
$c - a + b = 0$

Although 28 percent of the students correctly selected the first option as true, only 19 percent selected *only* the first option and *not* the second. Nine percent of the students selected both options as true. Interestingly, for both of these last two pattern analyses, there was no significant effect of which class students are in on their ability to produce the correct pattern of responses: Elementary Algebra students were no more successful than Pre-Algebra or Arithmetic students. This is a very intriguing result. It suggests that students who place into algebra may not really differ all that much in terms of their conceptual understanding from students placed into basic arithmetic or pre-algebra classes. The main difference may simply be in the ability to correctly remember and execute procedures, a kind of knowledge that is fragile without a deeper conceptual understanding of fundamental mathematical ideas.

In fact, none of the other items presented in this section showed significant differences in performance across the different classes. Clearly, there must be something different across these three classes of students - hence their placement into the different classes. Yet, in terms of reasoning and understanding in the context of non-standard questions, we could find few differences.

For the next two questions we told students the answer, but asked them to explain why it must be true. The first question was, "Given that x is a real number, neither of these equations has a real solution. Can you explain why that would be the case?" The equations were:

$x + 1 = x$

$x^2 = -9$

Forty-seven percent of the students could not think of any explanation for why there would be no real solution to the first equation. For the second equation, 50 percent could not generate an explanation. An additional 8 percent of students for the first equation (7 percent for the second equation) said that it would not be possible to know if the equations were true or not unless they could know what x is.

For the first equation, 23 percent of students tried to solve it with an algebraic manipulation. For example, they started with $x + 1 = x$, subtracted x from both sides, and then wrote down on their paper $1 = 0$. Or, they subtracted 1 from each side and wrote: $x = x - 1$. Once they had obtained these results they did not know what to do or say next. Similarly, for the second equation, 20 percent launched into an algebraic manipulation. Starting with $x^2 = -9$, for example, these students tried taking the square root of both sides, subtracting x from both sides, and so on.

Only 10 percent of students were able to give a good explanation for the first equation, and only 9 percent for the second equation. For the first equation, these correct explanations included: "Because if you add 1 to anything or any number, the answer has to be different than the letter in the question or equation;" or, "x can't equal itself + 1." For the second equation, correct explanations included: "A squared number should be positive since the first number was multiplied by itself;" or, "not possible because positive times positive will always be positive and negative times negative will always be positive."

Two more questions help to round out our exploration of students' reasoning about quantitative relations. The first provided the equation $x - a = 0$ and asked, "Assuming a is positive, if a increases, x would: (a) increase, (b) decrease, (c) remain the same, or (d) can't tell. Only 25 percent of students correctly chose increase. Thirty-four percent chose decrease, 23 percent, remain the same, and 11 percent said that you can't tell. The second provided the equation $ax = 1$ and asked the same question. Only 15 percent of students got this item correct (decrease). Thirty-two percent said increase, 33 percent, remain the same, and 14 percent said that you can't tell.

As with the previous items in this section, there was no significant difference in performance between students taking Arithmetic, Pre-Algebra, and Algebra.

Conclusions

In this study we drew from two sources of data to paint a picture of what community college developmental mathematics students know about mathematics. Three findings are particularly worthy of note and jointly they have implications for the kind of instruction that may prove beneficial for community college students. First, our examination of responses on the MDTP showed that the most common errors made by students were made whenever the possibility to do so was present. Students' routines for calling upon procedures to solve problems appears to be well-established. Although their knowledge of mathematical concepts may be fragile, their knowledge of procedures is firmly rooted—albeit in faulty notions of when and how procedures should be applied. Second, students can apply appropriate reasoning under the right conditions, but that form of knowledge is rarely accessed. Finally, when students are able to provide conceptual explanations, they also produce correct answers. (Whether the relationship is causal is deserving of further exploration.) Together, the results suggest that students should be encouraged to draw more extensively on their extant conceptual reasoning and, where necessary, be provided with the skills to do so. Instructors should be prepared though, to aid students in breaking flawed procedural habits, many of which have spent years in the making. We argue here that these suggestions for instruction could improve learning outcomes for developmental students, but our findings beg the question: might these students' difficulties in math have been prevented if the same suggestions were to have been applied to their earlier grades?

Our findings about students' thinking suggest that there are positive consequences to understanding mathematical concepts. Together with prior research on K-12 teaching practices, our findings are also consistent with the hypothesis that there are long-term, negative consequences of an almost exclusive focus on teaching mathematics as a large number of procedures that must be remembered, step-by-step, over time. As the number of procedures to be remembered grows – as it does through the K-12 curriculum – it becomes harder and harder for most students to remember them. Perhaps most disturbing is that the students in community college developmental mathematics courses did, for the most part, pass high school algebra. They were able, at one point, to remember enough to pass the tests they were given in high school. But as they moved into community college, many of the procedures were forgotten, or partly forgotten, and the fragile nature of their knowledge is revealed. Because the procedures were never connected with conceptual understanding of fundamental mathematics concepts, they have little to fall back on when the procedures fade.

The placement tests we examined provide ample evidence that students entering community colleges have difficulty with the procedures of mathematics. What is suggested by our data is that the reason for these procedural difficulties might be tied to a condition we are calling conceptual atrophy: students enter school with basic intuitive ideas about mathematics. They know, for example, that when you combine two quantities you get a larger quantity, that when you take half of something you get a smaller quantity. But because our educational practices have not tried to connect these intuitive ideas to mathematical notation and mathematical procedures, the willingness and ability to bring reason to bear on mathematical problems lies dormant. The fact that the community college students have so much difficulty with mathematical notation is significant, for mathematical notation plays a major part in mathematical reasoning. Because these students have not been asked to reason, they also have not needed the rigors of mathematical notation, and so have not learned it.

Though we have learned a great deal from data reported here, we remain unsatisfied with the richness of the picture we've painted and our work on the topic therefore continues. We alluded earlier to our intent to carry out one-on-one interviews with developmental mathematics students. Those interviews will be used to dig deeper in each case, trying to discern what, precisely, underlies each student's difficulties with mathematics. Is it simply failure to remember the conventions of mathematics? Is it a deficiency in basic knowledge of number and quantity? Is it a lack of conceptual understanding? What do these students understand about basic mathematics concepts (regardless of their ability to solve school-like mathematics problems)? Beyond this, we have two goals unique to the interviews. Through them we will examine what students think it means to do mathematics. Is is just remembering, or is reasoning also required? Finally, we will examine whether students can reason if provided an opportunity and pressed to do it. Can they discover some new mathematical fact based only on making effective use of other facts they know?

A full report of the results of the interviews will be forthcoming, but there is already some good news to share. We have found that it is often possible to coax students into reasoning by first asking them questions that could be answered by reasoning, and second, by giving them permission to reason (instead of doing it the way they were taught). Furthermore, the thirty students we have interviewed uniformly find the interview interesting, even after spending well over an hour with the interviewer, thinking hard about fundamental mathematics concepts. This gives us further cause to believe that developmental math students might respond well to a reason-focused mathematics class in which they are given opportunities to reason, and tools to support their reasoning.

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Lucky Larry

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This ingenious solution was provided as a test solution by one of my students, who seems to be on a mission to stamp out the distributive rule. His process seems to work well on any proportion with number patterns similar to this example.

Solve: $\dfrac{2x-5}{12} = \dfrac{5x-2}{12}$

$$\frac{2x-5}{12} = \frac{5x-2}{12}$$
$$2x - 5 \cdot 12 = 12 \cdot 5x - 2$$
$$2x - 60 = 60x - 2$$
$$-60 = 58x - 2$$
$$-58 = 58x$$
$$x = -1$$